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GENERALISED RELATIVE FIX POINTS OF k-ITERATED FUNCTIONS OF CLASS II

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Abstract: Introducing the idea of generalised relative iterations of k functions of class II, we extend a theorem on fix point involving exact order.

Keywords and Phrases: Generalised iteration, relative fix point, class II functions.

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1. Introduction and Definitions

Let $f(z)$ be a single valued function of the complex variable z. Then $f(z)$ is said to belong to (i) class I if $f(z)$ is entire transcendental, (ii) class II if it is regular in the complex plane punctured at $a, b\,(a \neq b)$ and has an essential singularity at b and a singularity at a and if $f(z)$ does not assume the values a and b anywhere in the complex plane except possible at the point a.

For simplicity we take $a = 0$ and $b = +\infty$.

The functions $f_n(z)$ of $f(z)$ are defined by

$$f_0(z) = z \text{ and } f_{n+1}(z) = f(f_n(z)) \text{ for } n = 0, 1, 2, ... \qquad .$$

Definition 1.1. *A point α is called a fix point of $f(z)$ of order n if α is a solution of $f_n(z) = z$ and called a fix point of exact order n if α is a solution of $f_n(z) = z$ but not a solution of $f_k(z) = z, k = 1, 2, ..., n-1$.*

Regarding the existence of a fix point, Baker [1] proved the following theorem.

Theorem 1.2. [1] *If $f(z)$ belongs to class I, then $f(z)$ has fix points of exact order n, except for at most one value of n.*

Then Bhattacharyya [2] extended the above theorem for the functions of class II.

Theorem 1.3. [2] *If $f(z)$ belongs to class II, then $f(z)$ has infinitely many fix points of exact order n, for every positive integer n.*

After this in [5], Lahiri and Banerjee introduced the concept of relative iteration defined as follows.

Let f and g be functions of the complex variable z.

$$\begin{aligned} Let \quad f_1 &= f \\ f_2 &= f \circ g = f \circ g_1 \\ f_3 &= f \circ g \circ f = f \circ g_2 \\ &\vdots \\ f_n &= f \circ g \circ f \circ g \circ ... \circ f \text{ or } g \text{ according} \\ &\quad \text{as } n \text{ is odd or even} \\ &= f \circ g_{n-1}. \end{aligned}$$

Similarly

$$\begin{aligned} g_1 &= g \\ g_2 &= g \circ f = g \circ f_1 \\ g_3 &= g \circ f \circ g = g \circ f_2 \\ &\vdots \\ g_n &= g \circ f_{n-1}. \end{aligned}$$

Here all f_n and g_n are functions in class II, if f and g are so.

Definition 1.4. *A point β is called a fix point of $f(z)$ of order n with respect to $g(z)$, if $f_n(\beta) = \beta$ and a fix point of exact order n if $f_n(\beta) = \beta$ but $f_k(\beta) \neq \beta, k = 1, 2, 3, ..., n-1$. These points β are also called relative fix points.*

Theorem 1.5. [5] *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has infinitely many* *relative fix points of exact order n for every positive integer n, provided $\frac{T(r,g_n)}{T(r,f_n)}$ is bounded, where $T(r,f_n)$ and $T(r,g_n)$ are Nevanlinna's characteristic function for f_n and g_n respectively.*

Now we consider k non-constant functions $f_1, f_2, ..., f_k$ of the complex variable z and $0 < \alpha \leq 1$. We form the generalised iterations as follows.

$$\begin{aligned} F_1^1 &= (1-\alpha)z + \alpha f_1 \\ F_2^1 &= (1-\alpha)F_1^2 + \alpha\left(f_1 \circ F_1^2\right) \\ F_3^1 &= (1-\alpha)F_2^2 + \alpha\left(f_1 \circ F_2^2\right) \\ &\vdots \\ F_n^1 &= (1-\alpha)F_{n-1}^2 + \alpha\left(f_1 \circ F_{n-1}^2\right) \quad . \end{aligned}$$

Similarly

$$\begin{aligned} F_1^2 &= (1-\alpha)z + \alpha f_2 \\ F_2^2 &= (1-\alpha)F_1^3 + \alpha\left(f_2 \circ F_1^3\right) \\ F_3^2 &= (1-\alpha)F_2^3 + \alpha\left(f_2 \circ F_2^3\right) \\ &\vdots \\ F_n^2 &= (1-\alpha)F_{n-1}^3 + \alpha\left(f_2 \circ F_{n-1}^3\right) \end{aligned}$$

and

$$\begin{aligned} F_1^k &= (1-\alpha)z + \alpha f_k \\ F_2^k &= (1-\alpha)F_1^1 + \alpha\left(f_k \circ F_1^1\right) \\ F_3^k &= (1-\alpha)F_2^1 + \alpha\left(f_k \circ F_2^1\right) \\ &\vdots \\ F_n^k &= (1-\alpha)F_{n-1}^1 + \alpha\left(f_k \circ F_{n-1}^1\right) \quad . \end{aligned}$$

Here all F_n^1, F_n^2, ..., F_n^k are functions of class II if f_i; $i = 1, 2, ..., k$ are so.

Now we introduce the following definition.

Definition 1.6. *A point β is called a generalised fix point of $f_1(z)$ of order n with respect to $f_2(z)$, $f_3(z)$, ..., $f_k(z)$ if $F_n^1(\beta) = \beta$ and a generalised fix point of $f_1(z)$ of exact order n with respect to $f_2(z)$, $f_3(z)$, ..., $f_k(z)$ if $F_n^1(\beta) = \beta$ but $F_s^1(\beta) \neq \beta$; $s = 1, 2, ..., n-1$. These points are called generalised relative fix points.*

Example 1.7. Let $f_1(z) = 2z+1$, $f_2(z) = 2z+2$, $f_3(z) = 2z+3$, $f_4(z) = 2z+4$. Also choose $\alpha = \frac{1}{2}$.

Then

$$\begin{aligned} F_1^4(z) &= (1-\alpha)\, z + \alpha f_4(z) \\ &= \frac{1}{2} z + \frac{1}{2}(2z+4) \\ &= \frac{1}{2}(3z+4), \end{aligned}$$

$$\begin{aligned} F_2^3(z) &= (1-\alpha)\, F_1^4(z) + \alpha f_3\left(F_1^4(z)\right) \\ &= \frac{1}{2}\left(\frac{3z+4}{2}\right) + \frac{1}{2}\left(2.\frac{3z+4}{2} + 3\right) \\ &= \frac{9(z+2)}{4}, \end{aligned}$$

$$\begin{aligned} F_3^2(z) &= (1-\alpha)\, F_2^3(z) + \alpha f_2\left(F_2^3(z)\right) \\ &= \frac{1}{2}.\frac{9(z+2)}{4} + \frac{1}{2}\left(2.\frac{9(z+2)}{4} + 2\right) \\ &= \frac{27z+62}{8}, \end{aligned}$$

and

$$\begin{aligned} F_4^1(z) &= (1-\alpha)\, F_3^2(z) + \alpha f_1\left(F_3^2(z)\right) \\ &= \frac{1}{2}.\frac{27z+62}{8} + \frac{1}{2}\left(2.\frac{27z+62}{8} + 1\right) \\ &= \frac{81z+194}{16}. \end{aligned}$$

Now

$$\begin{aligned} F_4^1(z) &= z \\ \text{implies} \quad \frac{81z+194}{16} &= z \\ \text{implies} \quad z &= -\frac{194}{65}. \end{aligned}$$

Again

$$\begin{aligned} F_1^3(z) &= (1-\alpha)\, z + \alpha f_3(z) \\ &= \frac{1}{2}.z + \frac{1}{2}(2z+3) \\ &= \frac{3}{2}(z+1), \end{aligned}$$

$$\begin{aligned} F_2^2(z) &= (1-\alpha)\, F_1^3(z) + \alpha f_2\left(F_1^3(z)\right) \\ &= \frac{1}{2}.\frac{3(z+1)}{2} + \frac{1}{2}\left\{2.\frac{3(z+1)}{2} + 2\right\} \\ &= \frac{9z+13}{4}, \end{aligned}$$

$$\begin{aligned} F_3^1(z) &= (1-\alpha)\, F_2^2(z) + \alpha f_1\left(F_2^2(z)\right) \\ &= \frac{1}{2}.\frac{9z+13}{4} + \frac{1}{2}\left(2.\frac{9z+13}{4} + 1\right) \\ &= \frac{27z+43}{8}. \end{aligned}$$

Now

$$\begin{aligned} F_3^1(z) &= z \\ \text{implies} \quad \frac{27z+43}{8} &= z \\ \text{implies} \quad z &= -\frac{43}{19}. \end{aligned}$$

Similarly

$$\begin{aligned} F_1^2(z) &= (1-\alpha)\, z + \alpha f_2(z) \\ &= \frac{1}{2}.z + \frac{1}{2}(2z+2) \\ &= \frac{1}{2}(3z+2), \end{aligned}$$

$$\begin{aligned} F_2^1(z) &= (1-\alpha)\, F_1^2(z) + \alpha f_1\left(F_1^2(z)\right) \\ &= \frac{1}{2}.\frac{3z+2}{2} + \frac{1}{2}\left(2.\frac{3z+2}{2} + 1\right) \\ &= \frac{9z+8}{4}. \end{aligned}$$

Now

$$\begin{aligned} F_2^1(z) &= z \\ \text{implies} \quad \frac{9z+8}{4} &= z \\ \text{implies} \quad z &= -\frac{8}{5}. \end{aligned}$$

And

$$\begin{aligned} F_1^1(z) &= (1-\alpha)z + \alpha f_1(z) \\ &= \frac{1}{2}.z + \frac{1}{2}(2z+1) \\ &= \frac{1}{2}(3z+1). \end{aligned}$$

Finally

$$\begin{aligned} F_1^1(z) &= z \\ \text{implies} \quad \frac{3z+1}{2} &= z \\ \text{implies} \quad z &= -1. \end{aligned}$$

Therefore $z = -\frac{194}{65}$ is a fix point of $f_1(z)$ of exact order 4.

Let $f(z)$ be meromorphic in $r_0 \le |z| < +\infty$, $r_0 > 0$. Here we use the following notations.

$$\begin{aligned} n(t,a,f) &= \text{the number of roots of } f(z) = a \text{ in } r_0 < |z| \le t, \text{counted according} \\ &\quad \text{to multiplicity}, \\ N(r,a,f) &= \int_{r_0}^{r} \frac{n(t,a,f)}{t} dt, \\ n(t,\infty,f) &= n(t,f) = \text{number of poles of } f(z) \text{ in } r_0 < |z| \le t, \text{ counted} \\ &\quad \text{with due to multiplicity} \\ N(r,\infty,f) &= N(r,f), \\ m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta , \\ m(r,a,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta}) - a}\right| d\theta \text{ and} \\ T(r,f) &= m(r,f) + N(r,f). \end{aligned}$$

From first fundamental theorem we have

$$m(r,a,f) + N(r,a,f) = T(r,f) + O(\log r), \tag{1.1}$$

where $r_0 \le |z| < +\infty$, $r_0 > 0$.

We now suppose that $f(z)$ is non-constant. Let $a_{1,}a_{2,\dots,}a_q; q \ge 2$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\upsilon| \ge \delta$ for $1 \le \mu < \upsilon \le q$.

Then

$$m(r,f)+\sum_{\upsilon=1}^{q}m(r,a_{\upsilon},f)\leq 2T(r,f)-N_{1}(r)+S(r), \tag{1.2}$$

where

$$N_{1}(r)=N\left(r,\frac{1}{f'}\right)+2N(r,f)-N(r,f'),$$

and

$$S(r)=m\left(r,\frac{f'}{f}\right)+\sum_{\upsilon=1}^{q}m\left(r,\frac{f'}{f-a_{\upsilon}}\right)+O(\log r).$$

Adding $N(r,f)+\sum_{\upsilon=1}^{q}N(r,a_{\upsilon},f)$ to both sides of (1.2) and using (1.1), we obtain

$$(q-1)T(r,f)\leq \overline{N}(r,f)+\sum_{\upsilon=1}^{q}\overline{N}(r,a_{\upsilon},f)+S_{1}(r), \tag{1.3}$$

where $S_{1}(r)=O(\log T(r,f))$ and $\overline{N}$ corresponds to distinct roots.

Again if f_n has an essential singularity at ∞, we have $\frac{\log r}{T(r,f_n)}\to 0$ as $r\to+\infty$.

2. Lemmas

To prove the main result we need the following lemmas.

Lemma 2.1. *If f and g are functions in class II, then for any $r_0>0$ and M, a positive constant $\frac{T(r,f\circ g)}{T(r,g)}>M$ for all large r, except a set of r intervals of total finite length.*

This follows from a lemma of [5] simply by taking $n=1$ and $p=1$.

Lemma 2.2. *If n is any positive integer and $f_1,f_2,...,f_k$ are functions in class II, then for any $r_0>0$ and a suitable positive constant M_1, we have*

$$\frac{T\left(r,F_{n+p}^{1}\right)}{T\left(r,F_{n}^{1}\right)}>M_1 \quad or \quad \frac{T\left(r,F_{n+p}^{2}\right)}{T\left(r,F_{n}^{1}\right)}>M_1 \quad or \ ..... \ or \quad \frac{T\left(r,F_{n+p}^{k}\right)}{T\left(r,F_{n}^{1}\right)}>M_1$$

according as $p=km$ or $km-1$ or...or $km-(k-1)$, $m\in\mathbb{N}$ for all large r, except a set of r intervals of total finite length.

Proof. For $j=1,2,...,n$ and for all large r, by using Lemma 2.1, we get

$$\begin{aligned}T\left(r,F_{j+1}^{1}\right)&\leq T\left(r,(1-\alpha)F_{j}^{2}\right)+T\left(r,\alpha f_1\circ F_{j}^{2}\right)+O(1)\\&\leq T\left(r,F_{j}^{2}\right)+T\left(r,f_1\circ F_{j}^{2}\right)+O(1)\\&=T\left(r,f_1\circ F_{j}^{2}\right)\left[1+\frac{T\left(r,F_{j}^{2}\right)}{T\left(r,f_1\circ F_{j}^{2}\right)}+\frac{O(1)}{T\left(r,f_1\circ F_{j}^{2}\right)}\right]\\&=(1+O(1))\,T\left(r,f_1\circ F_{j}^{2}\right)\quad .\end{aligned} \tag{2.1}$$

Again $f_1 \circ F_j^2 = \frac{1}{\alpha} F_{j+1}^1 - \frac{1-\alpha}{\alpha} F_j^2$and so for large r

$$T\left(r, f_1 \circ F_j^2\right) \leq T\left(r, F_{j+1}^1\right) + T\left(r, F_j^2\right) + O(1) \quad .$$

Therefore

$$\begin{aligned} T\left(r, F_{j+1}^1\right) &\geq T\left(r, f_1 \circ F_j^2\right) - T\left(r, F_j^2\right) + O(1) \\ &= T\left(r, f_1 \circ F_j^2\right)\left[1 - \frac{T\left(r, F_j^2\right)}{T\left(r, f_1 \circ F_j^2\right)} + \frac{O(1)}{T\left(r, f_1 \circ F_j^2\right)}\right] \\ &= (1 + O(1))\, T\left(r, f_1 \circ F_j^2\right) \quad . \end{aligned} \tag{2.2}$$

From (2.1) and (2.2) for all large r, we have

$$T\left(r, F_{j+1}^1\right) = (1 + O(1))\, T\left(r, f_1 \circ F_j^2\right) \quad .$$

Similarly for large r, we have

$$T\left(r, F_{j+1}^2\right) = (1 + O(1))\, T\left(r, f_2 \circ F_j^3\right)$$

$$\vdots$$

and so in general

$$T\left(r, F_{j+1}^k\right) = (1 + O(1))\, T\left(r, f_k \circ F_j^1\right) \quad . \tag{2.3}$$

First suppose $p = km, m \in \mathbb{N}$.

Then for all large r except a set of r intervals of total finite length, we have from (2.3) and by using Lemma 2.1

$$\begin{aligned} \frac{T\left(r, F_{n+p}^1\right)}{T\left(r, F_n^1\right)} &= (1 + O(1)) \frac{T\left(r, f_1 \circ F_{n+p-1}^2\right)}{T\left(r, F_n^1\right)} \\ &= (1 + O(1)) \frac{T\left(r, f_1 \circ F_{n+p-1}^2\right)}{T\left(r, F_{n+p-1}^2\right)} \frac{T\left(r, F_{n+p-1}^2\right)}{T\left(r, F_n^1\right)} \\ &= (1 + O(1)) \frac{T\left(r, f_1 \circ F_{n+p-1}^2\right)}{T\left(r, F_{n+p-1}^2\right)} \frac{(1 + O(1)) T\left(r, f_2 \circ F_{n+p-2}^3\right)}{T\left(r, F_n^1\right)} \\ &= (1 + O(1)) \frac{T\left(r, f_1 \circ F_{n+p-1}^2\right)}{T\left(r, F_{n+p-1}^2\right)} \frac{T\left(r, f_2 \circ F_{n+p-2}^3\right)}{T\left(r, F_{n+p-2}^3\right)} \frac{T\left(r, F_{n+p-2}^3\right)}{T\left(r, F_n^1\right)} \\ &\vdots \\ &= (1 + O(1)) \frac{T\left(r, f_1 \circ F_{n+p-1}^2\right)}{T\left(r, F_{n+p-1}^2\right)} \frac{T\left(r, f_2 \circ F_{n+p-2}^3\right)}{T\left(r, F_{n+p-2}^3\right)} \frac{T\left(r, f_3 \circ F_{n+p-3}^4\right)}{T\left(r, F_{n+p-3}^4\right)} \end{aligned}$$

$$
\begin{aligned}
& \cdots \frac{T\left(r, f_k \circ F_n^1\right)}{T\left(r, F_n^1\right)} \\
> & \ (1+O(1)) M^p \\
= & \ M_1 \text{ say, where } M_1=(1+O(1)) M^p, \text{ a positive constant}
\end{aligned}
$$

i.e,

$$
\frac{T\left(r, F_{n+p}^1\right)}{T\left(r, F_n^1\right)}>M_1
$$

for all large r except a set of r intervals of total finite length.

Next suppose $p=km-1, m \in \mathbb{N}$.

Then for all large r except a set of r intervals of total finite length, we have from (2.3) and by using Lemma 2.1

$$
\begin{aligned}
\frac{T\left(r, F_{n+p}^2\right)}{T\left(r, F_n^1\right)} & = (1+O(1)) \frac{T\left(r, f_2 \circ F_{n+p-1}^3\right)}{T\left(r, F_n^1\right)} \\
& = (1+O(1)) \frac{T\left(r, f_2 \circ F_{n+p-1}^3\right)}{T\left(r, F_{n+p-1}^3\right)} \frac{T\left(r, F_{n+p-1}^3\right)}{T\left(r, F_n^1\right)} \\
& = (1+O(1)) \frac{T\left(r, f_2 \circ F_{n+p-1}^3\right)}{T\left(r, F_{n+p-1}^3\right)} \frac{(1+O(1)) T\left(r, f_3 \circ F_{n+p-2}^4\right)}{T\left(r, F_n^1\right)} \\
& = (1+O(1)) \frac{T\left(r, f_2 \circ F_{n+p-1}^3\right)}{T\left(r, F_{n+p-1}^3\right)} \frac{T\left(r, f_3 \circ F_{n+p-2}^4\right)}{T\left(r, F_{n+p-2}^4\right)} \frac{T\left(r, F_{n+p-2}^4\right)}{T\left(r, F_n^1\right)} \\
& \vdots \\
& = (1+O(1)) \frac{T\left(r, f_2 \circ F_{n+p-1}^3\right)}{T\left(r, F_{n+p-1}^3\right)} \frac{T\left(r, f_3 \circ F_{n+p-2}^4\right)}{T\left(r, F_{n+p-2}^4\right)} \frac{T\left(r, f_4 \circ F_{n+p-3}^5\right)}{T\left(r, F_{n+p-3}^5\right)} \\
& \quad \cdots \frac{T\left(r, f_k \circ F_n^1\right)}{T\left(r, F_n^1\right)} \\
& > (1+O(1)) M^p \\
& = M_1 \text{ say, where } M_1=(1+O(1)) M^p, \text{ a positive constant}
\end{aligned}
$$

i.e,

$$
\frac{T\left(r, F_{n+p}^2\right)}{T\left(r, F_n^1\right)}>M_1
$$

for all large r except a set of r intervals of total finite length.

Finally suppose $p=km-(k-1), m \in \mathbb{N}$.

Then for all large except a set of r interval of total finite length, we have from (2.3) and by using Lemma 2.1

$$\begin{aligned}
\frac{T\left(r,F_{n+p}^{k}\right)}{T\left(r,F_{n}^{1}\right)} &= (1+O(1))\frac{T\left(r,f_k\circ F_{n+p-1}^{1}\right)}{T\left(r,F_{n}^{1}\right)}\\
&= (1+O(1))\frac{T\left(r,f_k\circ F_{n+p-1}^{1}\right)}{T\left(r,F_{n+p-1}^{1}\right)}\frac{T\left(r,F_{n+p-1}^{1}\right)}{T\left(r,F_{n}^{1}\right)}\\
&= (1+O(1))\frac{T\left(r,f_k\circ F_{n+p-1}^{1}\right)}{T\left(r,F_{n+p-1}^{1}\right)}\frac{(1+O(1))T\left(r,f_1\circ F_{n+p-2}^{2}\right)}{T\left(r,F_{n}^{1}\right)}\\
&= (1+O(1))\frac{T\left(r,f_k\circ F_{n+p-1}^{1}\right)}{T\left(r,F_{n+p-1}^{1}\right)}\frac{T\left(r,f_1\circ F_{n+p-2}^{2}\right)}{T\left(r,F_{n+p-2}^{2}\right)}\frac{T\left(r,F_{n+p-2}^{2}\right)}{T\left(r,F_{n}^{1}\right)}\\
&\;\;\vdots\\
&= (1+O(1))\frac{T\left(r,f_k\circ F_{n+p-1}^{1}\right)}{T\left(r,F_{n+p-1}^{1}\right)}\frac{T\left(r,f_1\circ F_{n+p-2}^{2}\right)}{T\left(r,F_{n+p-2}^{2}\right)}\frac{T\left(r,f_2\circ F_{n+p-3}^{3}\right)}{T\left(r,F_{n+p-3}^{3}\right)}\\
&\quad \ldots\frac{T\left(r,f_k\circ F_{n}^{1}\right)}{T\left(r,F_{n}^{1}\right)}\\
&> (1+O(1))M^{p}\\
&= M_1 \text{ say, where } M_1=(1+O(1))M^{p},\ \text{a positive constant}
\end{aligned}$$

i.e,

$$\frac{T\left(r,F_{n+p}^{k}\right)}{T\left(r,F_{n}^{1}\right)} > M_1$$

for all large r except a set of r intervals of total finite length.

Lemma 2.3. *If n is any positive integer and $f_1, f_2, ..., f_k$ are functions in class II, then for any $r_0 > 0$ and a suitable positive constant M_1, we have*

$$\frac{T\left(r,F_{n+p}^{2}\right)}{T\left(r,F_{n}^{2}\right)} > M_1 \quad or \quad \frac{T\left(r,F_{n+p}^{3}\right)}{T\left(r,F_{n}^{2}\right)} > M_1 \quad or \;.....\; or \quad \frac{T\left(r,F_{n+p}^{1}\right)}{T\left(r,F_{n}^{2}\right)} > M_1$$

according as $p = km$ or $km-1$ or...or $km-(k-1)$, $m \in \mathbb{N}$ for all large r, except a set of r intervals of total finite length.

Lemma 2.4. *If n is any positive integer and $f_1, f_2, ..., f_k$ are functions in class II, then for any $r_0 > 0$ and a suitable positive constant M_1, we have*

$$\frac{T\left(r,F_{n+p}^{k}\right)}{T\left(r,F_{n}^{k}\right)} > M_1 \quad or \quad \frac{T\left(r,F_{n+p}^{1}\right)}{T\left(r,F_{n}^{k}\right)} > M_1 \quad or \;.....\; or \quad \frac{T\left(r,F_{n+p}^{k-1}\right)}{T\left(r,F_{n}^{k}\right)} > M_1$$

according as $p = km$ *or* $km - 1$ *or...or* $km - (k-1)$, $m \in \mathbb{N}$ *for all large* r, *except a set of* r *intervals of total finite length.*

3. Main Result

Our main result is the following theorem.

Theorem 3.1. *If* f_1, f_2, ..., f_k *belong to class II, then* $f_1(z)$ *has an infinity of generalised relative fix points of exact order* n $(> k)$ *for every positive integer* n, *provided* $\frac{T(r,F_n^i)}{T(r,F_n^1)}$, $i = 2, 3, ..., k$ *are bounded.*

Proof. Here we consider the function

$$g(z) = \frac{F_n^1(z)}{z}, \; r_0 < |z| < +\infty .$$

Then

$$T(r,g) = T\left(r, F_n^1\right) + O(\log r). \tag{3.1}$$

Now we assume that $f_1(z)$ has only a finite number of generalised relative fix points of exact order n. We take $q = 2$, $a_1 = 0$, $a_2 = 1$, then from (1.3) we obtain

$$T(r,g) \le \overline{N}(r,\infty,g) + \overline{N}(r,0,g) + \overline{N}(r,1,g) + S_1(r,g), \tag{3.2}$$

where $S_1(r,g) = O(\log T(r,g))$ outside a set of r intervals of finite length .

We know that

$$\overline{N}(r,0,g) = \int_{r_0}^{r} \frac{\overline{n}(t,0,g)}{t} dt$$

where $\overline{n}(t,0,g)$ is the number of roots of $g(z) = 0$ in $r_0 < |z| \le t$, each multiple root taken once at a time. The distinct roots of $g(z) = 0$ in $r_0 < |z| \le t$ are the roots of $F_n^1(z) = 0$ in $r_0 < |z| \le t$. Now $F_n^1(z)$ has a singularity at $z = 0$, an essential singularity at $z = \infty$ and $F_n^1(z) \ne 0, \infty$. So $\overline{n}(t,0,g) = 0$ and so $\overline{N}(r,0,g) = 0$. Similarly $\overline{N}(r,\infty,g) = 0$. Thus (3.2) reduces to

$$T(r,g) \le \overline{N}(r,1,g) + S_1(r,g) \tag{3.3}$$

Again $F_n^1(z) = z$ when $g(z) = 1$.

Then

$$\begin{aligned}
\overline{N}(r,1,g) &= \overline{N}\left(r,0,F_n^1 - z\right) \\
&\le \sum_{j=1}^{n-2} \left[\overline{N}\left(r,0,F_j^1 - z\right) + \overline{N}\left(r,0,F_j^2 - z\right) + ... + \overline{N}\left(r,0,F_j^k - z\right)\right] \\
&\le \sum_{j=1}^{n-2} [T\left(r,F_j^1 - z\right) + O(1) + T\left(r,F_j^2 - z\right) + O(1) + ... \\
&\quad + T\left(r,F_j^k - z\right) + O(1)]
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^{n-2}[T\left(r,F_j^1\right)+T\left(r,F_j^2\right)+...+T\left(r,F_j^k\right)]+O\left(\log r\right)\\
&= [\{T\left(r,F_{j_1}^1\right)+T\left(r,F_{j_{k+1}}^1\right)+...+T\left(r,F_{j_{kp_1-(k-1)}}^1\right)\}+\{T\left(r,F_{j_2}^1\right)\\
&\quad +T\left(r,F_{j_{k+2}}^1\right)+...+T\left(r,F_{j_{kp_2-(k-2)}}^1\right)\}+...+\{T\left(r,F_{j_{k-1}}^1\right)+\\
&\quad T\left(r,F_{j_{2k-1}}^1\right)+...+T\left(r,F_{j_{kp_{k-1}-1}}^1\right)\}+\{T\left(r,F_{j_k}^1\right)+T\left(r,F_{j_{2k}}^1\right)\\
&\quad +...+T\left(r,F_{j_{kp_k}}^1\right)\}]+[\{T\left(r,F_{j_1}^2\right)+T\left(r,F_{j_{k+1}}^2\right)+...\\
&\quad +T\left(r,F_{j_{kp_1-(k-1)}}^2\right)\}+\{T\left(r,F_{j_2}^2\right)+T\left(r,F_{j_{k+2}}^2\right)...+T\left(r,F_{j_{kp_2-(k-2)}}^2\right)\}\\
&\quad +...+\{T\left(r,F_{j_{k-1}}^2\right)+T\left(r,F_{j_{2k-1}}^2\right)+...+T\left(r,F_{j_{kp_{k-1}-1}}^2\right)\}+\{T\left(r,F_{j_k}^2\right)\\
&\quad +T\left(r,F_{j_{2k}}^2\right)+...+T\left(r,F_{j_{kp_k}}^2\right)\}]+....+[\{T\left(r,F_{j_1}^k\right)+T\left(r,F_{j_{k+1}}^k\right)\\
&\quad +...+T\left(r,F_{j_{kp_1-(k-1)}}^k\right)\}+\{T\left(r,F_{j_2}^k\right)+T\left(r,F_{j_{k+2}}^k\right)+...+T\left(r,F_{j_{kp_2-(k-2)}}^k\right)\}\\
&\quad +...+\{T\left(r,F_{j_{k-1}}^k\right)+T\left(r,F_{j_{2k-1}}^k\right)+...+T\left(r,F_{j_{kp_{k-1}-1}}^k\right)\}+\{T\left(r,F_{j_k}^k\right)+\\
&\quad T\left(r,F_{j_{2k}}^k\right)+...+T\left(r,F_{j_{kp_k}}^k\right)\}]
\end{aligned}$$

where $j_1, j_{k+1}, ..., j_{kp_1-(k-1)}$; $j_2, j_{k+2}, ..., j_{kp_2-(k-2)}$; ...; $j_{k-1}, j_{2k-1}, ..., j_{kp_{k-1}-1}$; j_k, j_{2k}, ..., j_{kp_k} are strictly less than n and are of the form $kp_1-(k-1)$, $kp_2-(k-2)$, ..., $kp_{k-1}-1$, kp_k, $(p_1, p_2, .., p_{k-1}, p_k \in \mathbb{N})$

$$\begin{aligned}
&= T\left(r,F_n^1\right)[\{\frac{T\left(r,F_{j_k}^1\right)}{T\left(r,F_n^1\right)}+\frac{T\left(r,F_{j_{2k}}^1\right)}{T\left(r,F_n^1\right)}+...+\frac{T\left(r,F_{j_{kp_k}}^1\right)}{T\left(r,F_n^1\right)}\}+\{\frac{T\left(r,F_{j_1}^2\right)}{T\left(r,F_n^1\right)}\\
&\quad +\frac{T\left(r,F_{j_{k+1}}^2\right)}{T\left(r,F_n^1\right)}+...+\frac{T\left(r,F_{j_{kp_1-(k-1)}}^2\right)}{T\left(r,F_n^1\right)}\}+\{\frac{T\left(r,F_{j_2}^3\right)}{T\left(r,F_n^1\right)}+\frac{T\left(r,F_{j_{k+2}}^3\right)}{T\left(r,F_n^1\right)}\\
&\quad +...+\frac{T\left(r,F_{j_{kp_2-(k-2)}}^3\right)}{T\left(r,F_n^1\right)}\}+...+\{\frac{T\left(r,F_{j_{k-1}}^k\right)}{T\left(r,F_n^1\right)}+\frac{T\left(r,F_{j_{2k-1}}^k\right)}{T\left(r,F_n^1\right)}+...\\
&\quad +\frac{T\left(r,F_{j_{kp_{k-1}-1}}^k\right)}{T\left(r,F_n^1\right)}\}]\\
&\quad +T\left(r,F_n^2\right)[\{\frac{T\left(r,F_{j_{k-1}}^1\right)}{T\left(r,F_n^2\right)}+\frac{T\left(r,F_{j_{2k-1}}^1\right)}{T\left(r,F_n^2\right)}+...+\frac{T\left(r,F_{j_{kp_{k-1}-1}}^1\right)}{T\left(r,F_n^2\right)}\}
\end{aligned}$$

$$
\begin{aligned}
&+\{\frac{T\left(r,F^2_{j_k}\right)}{T\left(r,F^2_n\right)}+\frac{T\left(r,F^2_{j_{2k}}\right)}{T\left(r,F^2_n\right)}+...+\frac{T\left(r,F^2_{j_{kp_k}}\right)}{T\left(r,F^2_n\right)}\}+\{\frac{T\left(r,F^3_{j_1}\right)}{T\left(r,F^2_n\right)}+\frac{T\left(r,F^3_{j_{k+1}}\right)}{T\left(r,F^2_n\right)}\\
&+...+\frac{T\left(r,F^3_{j_{kp_1-(k-1)}}\right)}{T\left(r,F^2_n\right)}\}+...+\{\frac{T\left(r,F^k_{j_{k-2}}\right)}{T\left(r,F^2_n\right)}+\frac{T\left(r,F^k_{j_{2k-2}}\right)}{T\left(r,F^2_n\right)}+...\\
&+\frac{T\left(r,F^k_{j_{kp_{k-2}-2}}\right)}{T\left(r,F^2_n\right)}\}]+\text{...}\\
&...+T\left(r,F^k_n\right)[\{\frac{T\left(r,F^1_{j_1}\right)}{T\left(r,F^k_n\right)}+\frac{T\left(r,F^1_{j_{k+1}}\right)}{T\left(r,F^k_n\right)}+...+\frac{T\left(r,F^1_{j_{kp_1-(k-1)}}\right)}{T\left(r,F^k_n\right)}\}\\
&+\{\frac{T\left(r,F^2_{j_2}\right)}{T\left(r,F^k_n\right)}+\frac{T\left(r,F^2_{j_{k+2}}\right)}{T\left(r,F^k_n\right)}+...+\frac{T\left(r,F^2_{j_{kp_2-(k-2)}}\right)}{T\left(r,F^k_n\right)}\}+\{\frac{T\left(r,F^3_{j_{k-1}}\right)}{T\left(r,F^k_n\right)}\\
&+\frac{T\left(r,F^3_{j_{2k-1}}\right)}{T\left(r,F^k_n\right)}+...+\frac{T\left(r,F^3_{j_{kp_{k-1}-1}}\right)}{T\left(r,F^k_n\right)}\}+...+\{\frac{T\left(r,F^k_{j_k}\right)}{T\left(r,F^k_n\right)}+\frac{T\left(r,F^k_{j_{2k}}\right)}{T\left(r,F^k_n\right)}\\
&+...+\frac{T\left(r,F^k_{j_{kp_k}}\right)}{T\left(r,F^k_n\right)}\}]+O\left(\log r\right)\\
<\ &\frac{n-1}{2kn}T\left(r,F^1_n\right)+\frac{n-1}{2kn}T\left(r,F^2_n\right)+...+\frac{n-1}{2kn}T\left(r,F^k_n\right)+O\left(\log r\right),\\
&\text{using Lemma 2.2, Lemma 2.3 and Lemma 2.4.}
\end{aligned}
$$

So from (3.3) and since $\frac{T\left(r,F^2_n\right)}{T\left(r,F^1_n\right)}, \frac{T\left(r,F^3_n\right)}{T\left(r,F^1_n\right)}, ..., \frac{T\left(r,F^k_n\right)}{T\left(r,F^1_n\right)}$ are bounded, we have

$$
\begin{aligned}
T\left(r,g\right) &< \frac{n-1}{2kn}T\left(r,F^1_n\right)+\frac{n-1}{2kn}T\left(r,F^2_n\right)+...+\frac{n-1}{2kn}T\left(r,F^k_n\right)\\
&\quad +O\left(\log r\right)+S_1\left(r,g\right)\\
&= \frac{n-1}{2kn}T\left(r,F^1_n\right)+\frac{n-1}{2kn}T\left(r,F^2_n\right)+...+\frac{n-1}{2kn}T\left(r,F^k_n\right)\\
&\quad +O\left(\log r\right)+O\left(\log T\left(r,g\right)\right)\\
&\leq T\left(r,F^1_n\right)[\frac{n-1}{2kn}+\frac{n-1}{2kn}\frac{T\left(r,F^2_n\right)}{T\left(r,F^1_n\right)}+...+\frac{n-1}{2kn}\frac{T\left(r,F^k_n\right)}{T\left(r,F^1_n\right)}]\\
&\quad +\frac{O\left(\log\left(T\left(r,F^1_n\right)+O\left(\log r\right)\right)\right)}{T\left(r,F^1_n\right)}+\frac{O\left(\log r\right)}{T\left(r,F^1_n\right)}],\ \text{using (3.1)}\\
&\leq T\left(r,F^1_n\right)[\frac{n-1}{2kn}+\frac{n-1}{2kn}+...+\frac{n-1}{2kn}
\end{aligned}
$$

$$+\frac{O\left(\log\left(T\left(r,F_n^1\right)\left(1+\frac{O(\log r)}{T(r,F_n^1)}\right)\right)\right)}{T\left(r,F_n^1\right)}+\frac{O\left(\log r\right)}{T\left(r,F_n^1\right)}]$$
$$< T\left(r,F_n^1\right)[\frac{1}{2}+\frac{O\left(\log\left(T\left(r,F_n^1\right)\left(1+\frac{O(\log r)}{T(r,F_n^1)}\right)\right)\right)}{T\left(r,F_n^1\right)}+\frac{O\left(\log r\right)}{T\left(r,F_n^1\right)}]$$
$$= \frac{1}{2}T\left(r,F_n^1\right), \text{ for all large } r.$$

Therefore , $T(r,g) < \frac{1}{2}T\left(r,F_n^1\right)$ for all large r. This contradicts (3.1).

Hence $f_1(z)$ has infinitely many generalised relative fix points of exact order n.

This proves the theorem.

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